On Defining Mass

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Though central to any pedagogical development of physics, the concept of mass is still not well understood. Properly defining mass has proven to be far more daunting than contemporary textbooks would have us believe. And yet today the origin of mass is one of the most aggressively pursued areas of research in all of physics. Much of the excitement surrounding the Large Hadron Collider at CERN is associated with discovering the mechanism responsible for the masses of the elementary particles. This paper will first briefly examine the leading definitions, pointing out their shortcomings. Then, utilizing relativity theory, it will propose—for consideration by the community of physicists—a conceptual definition of mass predicated on the more fundamental concept of energy, more fundamental in that everything that has mass has energy, yet not everything that has energy has mass.

The usual definitions

Traditionally there have been three common approaches to defining mass: (1) as quantity of matter, (2) as that which resists changes in motion, and (3) as that which gives rise to the gravitational interaction. The first tack came out of the Middle Ages and its metaphysical musings.[1] The second goes back to Kepler (1618) and later to Newton's dynamics and still later to $\mathbf{F} = m\mathbf{a}$. The third evolved from the law of gravitation, $F = GmM/r^2$. The definitional failings of these approaches have been discussed in the literature[2] and will only be treated briefly here. Most of the conceptual difficulties arise when we try to integrate these classical definitions into the fabric of contemporary physics, illuminated as it is by the theory of relativity and quantum chromodynamics (QCD), the gauge theory of the strong interaction.

As a rule, when quantity of matter is proffered as the definiens of mass, it is linked to the "amount of stuff." But since the special theory tells us that a cannonball when heated gains rest energy and therefore mass, that approach is clearly problematic. The same number of the same atoms, now moving more violently, has more mass. As we'll see, most of the mass of ordinary matter in the universe comes from quark confinement energy, a phenomenon often lightheartedly referred to as "mass without mass." It follows that mass is not "quantity of matter" in any customary sense of the term, though it certainly has some relationship to it.

The second, more sophisticated definition appearing in countless textbooks and classrooms is based on the idea of inertia. *The mass of an object is a measure of, and gives rise to, its resistance to changes in motion*; $\mathbf{F} = m\mathbf{a}$, which stands on a rich experiential history, presumably quantifies the traditional idea of "inertia." Still, the rigorous statement of Newton's second law is in terms of the time rate of change of momentum $\mathbf{p}$; that is, $\mathbf{F} = d\mathbf{p}/dt$, not $\mathbf{F} = m\mathbf{a}$. The latter is the low speed approximation and m is generally not a proportionality constant between $\mathbf{F}$ and $\mathbf{a}$; these two vector quantities may not even be parallel. From relativity we have it that the relationship between $\mathbf{F}$ and $\mathbf{a}$ actually depends on the orientation of $\mathbf{F}$ with respect to $\mathbf{v}$; the velocity vector fixes a special direction in otherwise isotropic space. Acceleration doesn't necessarily occur along the line of action of the force.[3] When $\mathbf{F}$ and $\mathbf{v}$ are perpendicular, $\mathbf{F}_\perp = \gamma m \mathbf{a}_\perp$, and when $\mathbf{F}$ and $\mathbf{v}$ are parallel, $\mathbf{F}_{||} = \gamma^3 m \mathbf{a}_{||}$, where $\gamma = (1 - v^2/c^2)^{-1/2}$ and $\mathbf{p} = \gamma m \mathbf{v}$. If one still insists on defining "inertia" (call it m_I) in the usual classical way via $\mathbf{F} = m_\mathrm{I}\mathbf{a}$, then m_I must be a second-rank tensor, as compared with m, which is a scalar. That doesn't mean that mass and inertia are not connected concepts; they surely are. It just means there is no simple relationship that informs a straightforward definition of mass.

Consider a spaceship firing its constant-thrust engine; as it accelerates in the direction of $\mathbf{v}$, its speed increases, γ increases, and, even though $\mathbf{F}$ and m are constant, $\mathbf{a}$ continually decreases such that v never reaches c. Alternatively, if $\mathbf{a}$ is to be kept constant, $\mathbf{F}$ must increase as $\mathbf{v}$ increases. Only when $v \approx 0$ and $\gamma \approx 1$ will $m \approx F/a$. Increase the speed of an object and though its mass is constant, because of the nature of space-time (i.e., time dilation), it becomes harder and harder to sustain the acceleration; hence, the object's "inertia" increases. In other words, change the mass of a system and you change its inertia, but *the inertia of a linearly accelerating system changes even as its mass remains constant*. Or as Einstein and Infeld[4] put it: "If two bodies have the same rest mass, the one with the greater kinetic energy resists the action of an external force more strongly." Apparently mass is not identically inertia.

The third approach, less popular though common enough, is to be guided by $F = GmM/r^2$ and maintain that mass is that which gives rise to the gravitational interaction. But Newton's law of gravitation has been supplanted by the general theory of relativity. From that latter perspective what specifies gravity is the energy-momentum (or stress-energy) tensor. A moving object sails through space-time, which is curved by the presence of material matter. Its free motion is along a geodesic and corresponds to the straightest possible path, the one that follows the curvature of space-time. The object simply glides through space; there is no such thing as gravitational force.

According to John A. Wheeler, "A gravitational field is affected by mass-energy distributions and currents."[5] As before, mass is in there contributing, but not via force. It plays a central role although not in the simple way it did for Newton. On the other hand, quantum gravity reinstates the gravitational force as mediated by the exchange of gravitons, though as yet there is no adequate theory of precisely how that happens, and there is no direct evidence that it does. All of these complexities preclude formulating a rigorous definition of mass predicated on the gravitational interaction.

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Mass in terms of energy

Before we make use of $E = mc^2$, a related issue has to be settled. That equation can be misleading. As usually presented, E is the *total energy* of a system and $m = m_0(1 - v^2/c^2)^{-1/2}$ is its *relativistic mass*. That interpretation is quite problematic. Einstein himself never used "relativistic mass" in any of his scientific work. In the Einstein formulation of relativity, which we shall adhere to, m is speed independent and hence *invariant* (i.e., the same for all inertial observers; "invariant" does not mean unchangeable or constant). Moreover $E_0 = mc^2$, where E_0 is the rest energy (the energy of the system measured by a co-moving observer). When the system travels as a whole such that it has translational kinetic energy (KE), it follows that $E = E_0 + KE$ and $E \neq E_0$. Both E_0 and m are invariant. On the other hand, KE is not invariant, nor is $E = \gamma mc^2$.

To see how far-reaching the implications of the equation $E_0 = mc^2$ are, as regards our discussion of the nature of mass, consider a proton. It is composed of three nearly massless quarks (evidence suggests that the masses of the up and down quarks are a mere ≈ 4 MeV/c^2 and ≈ 8 MeV/c^2) exchanging massless gluons. Yet that tumultuous cloud of complex interactions manifests a considerable mass of 938.27 MeV/c^2, a mass now mostly attributed (via QCD) to its internal interaction energy, that is, its quark-confinement energy.[6] And the same is true for the neutron. Recent extraordinary calculations (Nov. 2008) using a procedure called lattice QCD were able to predict the masses of the nucleons and several light hadrons to within 4%.[7]

Since the mass of ordinary atomic matter—trees, airplanes, and us—is essentially nucleon mass, it originates primarily out of interactions, which result in rest energy, and hence mass. Consequently the mass of the standard kilogram is mostly quark-confinement energy. Or as Frank Wilczek put it in his Nobel lecture, "Asymptotic Freedom: From Paradox to Paradigm" (2004): "Most of the mass of ordinary matter—90% or more—arises from pure energy via $m = E/c^2$." And he went on to assert, "Einstein's law suggests the possibility of explaining mass in terms of energy. That is a good thing to do, because in modern physics energy is a more basic concept than mass."

Two fundamental equations of modern-day relativistic dynamics,

$$E^2 = m^2c^4 + \mathbf{p}^2c^2 \tag{1}$$

and

$$\mathbf{p}c^2 = E\mathbf{v}, \tag{2}$$

wherein $\mathbf{v}$ is velocity, can tell us a lot about mass.[8] Rearranging Eq. (1) leads to an expression for the mass of any particle:

$$m = (E^2 - \mathbf{p}^2c^2)^{1/2}/c^2. \tag{3}$$

Notice that for a single photon, $E = h\nu = pc$ and $m = 0$.

To formulate an equation for the invariant mass M of a system of particles (e.g., a nucleus or a cow) we make use of the additivity of both energy and momentum. Accordingly, consider the ith particle, which has energy E_i and momentum $\mathbf{p}_i$. Let the total energy and momentum of the system be E and $\mathbf{p}$, respectively, where

$$E = \Sigma_i E_i \qquad \text{and} \qquad \mathbf{p} = \Sigma_i \mathbf{p}_i\,. \tag{4}$$

Here the momentum of the system as a whole is the vector sum of the individual constituent contributions. It follows from Eqs. (3) and (4), summing over all the entities making up the system, that its total mass (which is not additive) is given by

$$M = (E^2 - \mathbf{p}^2c^2)^{1/2}/c^2. \tag{5}$$

The form of this equation matches Eq. (1), though the meaning of each term is somewhat different.

There are particles that have mass (like the proton) and perforce can exist at rest, and there are particles that are massless (like the photon) and cannot exist at rest. Consequently, the ideas of "mass" and "rest" are intimately related. For a system that possesses mass, an especially convenient reference frame can be found in which the net vector momentum of all the constituents is zero. We say that the system is at rest as viewed in that frame. Since $E = E_0 + KE$ and $E_0 = Mc^2$, $KE = E - Mc^2$ and Eq. (5) leads to

$$KE = (M^2c^4 + c^2\mathbf{p}^2)^{1/2} - Mc^2. \tag{6}$$

A system can expand or contract, or its components can fly around randomly and yet in its entirety it can be at rest. Just think of a motionless box containing a gas. Provided the net momentum is zero, Eq. (6) tells us that the translational KE of the system as a whole is zero; it is at rest, and $E = E_0$. Here E_0 is the net rest energy of the entire system; that is, the sum of the rest energies of each of the constituents, plus their individual internal kinetic energies, plus their potential energies due to any mutual interactions. It follows from Eq. (5), with $\mathbf{p} = 0$, that the net mass of the system is $M = E_0/c^2$. Since the rest energy of the whole is not the sum of only the rest energies of its separate parts, ***mass is not additive***. The mass of a system of objects, that each has mass, equals the sum of these individual masses only when they are all at rest with respect to each other, provided they do not interact. Still, ***mass is invariant***, and the system's mass determined in a zero-momentum frame is its mass in any other inertial frame.

The energy of a photon is entirely kinetic, $E = pc$; no zero-momentum frame exists for a single photon. That's also true for a collimated beam of photons, which has zero net mass. Yet a spray of equal-energy photons emerging isotropically from a motionless point source has a zero-momentum frame fixed at the center of the source. Taken as a whole, the expanding cloud of massless photons has a net rest energy, and a net mass. The same is true for a photon "gas" careening in every direction within a stationary oven, which itself demarcates a zero-momentum frame.[9]

Force, energy, matter, and mass

On the most fundamental level, there is matter, interaction, and change. All matter interacts and that interaction results in observable change. If matter did not interact, there would be no way to know that it even existed. We distinguish one sample of matter from another by the type and strength of its interactions. The four fundamental forces are known by the observable changes they produce; we do not "see" gravity, we see objects fall. To comprehend the world of experience, to link the fundamental interactions with that world, we invent quantities called "properties of matter" that are observable, or at least appear to be because they are readily deduced from observables. A bit of matter is apprehended by the perceivable effects of its various interactions. These behaviors (changes or resistance to changes) are associated with its basic properties (mass, charge, spin, etc.): no interactions, no properties—no properties, no physical existence. Consequently, it seems reasonable to suppose that an entity's properties derive their characteristics from its various interactions, internal and/or external.

Force is the agent of change;[10] it is that which causes change, be it a change in position overcoming some interaction, or a change in motion, or a transformation (e.g., of one quark into another). *Energy is a measure of change*.[11] Ordinarily we deal with processes occurring in existing systems, that is, with energy added to or removed from such systems. For example, the energy of a photon is made manifest by the observable change it produces upon being absorbed. Because energy is conserved, energy relates to change that has either already occurred or is yet to occur. In this we see an expression of the fact that energy and time are conjugate variables (as witnessed, e.g., by the uncertainty principle).

The increase or decrease in the energy of a system is a measure of the change (in whatever form it takes) in the state of that system due to any and all interactions it experiences. Framed in a more active way: ***energy is a measure of the ability of a system to produce change***. All matter, whether a single elementary particle or a composite system, possesses energy. An alteration in the energy of a system is an alteration in that system's capacity to undergo future self-generated change. For example, a stretched spring that is allowed to contract a little now, will be able to deliver less kinetic energy to a waiting pinball in the future, and thereby produce a lesser change in that ball's speed.

Matter comes in the form of "particles" and those particles individually manifest a variety of physical properties (such as mass, lifetime, wavelength, energy, charge, etc.). Along with Schrödinger, Born, de Broglie, Pauli, and others, we take photons to be the most ephemeral form of matter.[12] Thus we observe two different types of matter: matter that can exist at rest and matter that only exists in motion. For the sake of this discussion we can simply say that *matter is that which possesses determinable physical properties* or equivalently ***matter is that which interacts***. Some of these particle properties are always quantized, constant, and conserved (e.g., charge, spin, baryon number, strangeness, and charm), while others are not always quantized, constant, and/or conserved (e.g., mass, lifetime, energy, and momentum). Furthermore, all of the nonzero quantum numbers can be positive or negative (inclusive of antimatter). By contrast, mass is exclusively positive (for both matter and antimatter). Except for the photon, all "real" (as opposed to "virtual")[13] objects (e.g., quarks, leptons, and elephants) have mass. That universality is not true for any of the quantized properties: pions are spinless, neutrinos are chargeless, but each has mass. Mass is a very different kind of characteristic than is charge, or spin, or strangeness.

Mass is a property of matter that arises, in whole or in part, through interactions. At the present time it is widely, though not universally, assumed that the masses of the several elementary particles—quarks, leptons, Ws, and Z^0s—are imparted to those particles as a result of interactions with the Higgs field. If that proves to be the case, the masses of all the familiar things in our world arise entirely through interactions. In other words, the quarks and leptons get their masses by interacting with the fabled Higgs bosons, and neutrons and protons get their additional mass from quark-confinement energy. But of course, the Higgs field is still highly tentative and awaits verification. Alternatively there are those who believe that the mass of a body comes from its gravitational interaction with the rest of the universe with which it couples (via Mach's principle). This is appealing because it unites the two observed manifestations of mass, gravity and inertia, without introducing a new force. In any case, mass arises through interactions.

Since every particle can be annihilated, perhaps no portion of the mass of any entity is associated with "stuff" having intrinsic substantiality. That was essentially Einstein's position when he wrote: "It was found that inertia is not a fundamental property of matter [like charge or spin], nor indeed, an irreducible magnitude, but a property of energy."[14] Penrose and Rindler came to much the same conclusion from a different perspective: "This 'matter-like' behavior of energy strongly suggests that all of the rest-mass is due to internal energy, and that what we have called 'amount of matter' has no separate existence but is really a form of rest-energy."[15]

The relationship between interactions and mass is further underscored by the creation of mass in pair production. Though it often takes place through the action of the electromagnetic field, if Stephen Hawking is right, the intense gravitational field surrounding a black hole can also initiate pair production and lead to the creation of mass. On the other hand, the self-gravity of the Earth reduces its mass by a multiplicative factor of about 4.2×10^{-10} upon coming together as a sphere.[16] Apparently every kind of interaction can affect the mass of a system, be it a nucleon or a planet.

A modern definition of mass

Mindful of $E_0 = mc^2$, Eq. (5), and the fact that mass is a property of matter arising through interactions and is therefore associated with energy, we can at least begin to define mass from a contemporary perspective:

> The invariant mass of any object—elementary or composite—is a measure of the minimum amount of energy required to create that object, at rest, as it exists at that moment.

Overlooking the influence of temperature, it takes a minimum of about 10^{14} J to create 1.0 g of diamond, or 1.0 g of deuterium, or 1.0 g of daffodil; it doesn't matter how complex the structure of the object is or what it's composed of.

The above definition accords with the fact that a system consisting of a single photon has zero mass, since for it there is no frame in which it does not move. Indeed, a composite system in which every particle moves at c in the same direction is also [according to Eqs. (2) and (5)] massless. Because energy is conserved, *the mass of any object at rest is a measure of the maximum amount of kinetic energy that can be liberated in the process of annihilating that object.*

Imagine any physical process occurring in some isolated region. There are two complementary perspectives from which to apprehend what's happening. We could metaphorically stand "outside" and study the system as a whole, or we could stand "inside" and focus on the individual participants. When considered as a whole (i.e., from what we are calling "outside"), whatever *KE* there is, is internal *KE* and it contributes to the rest energy and therefore to the mass of the system. A cannonball is more massive hot than cold. And that comports with our definition since it takes more energy to create a hot cannonball at rest than an otherwise identical cold one.

By contrast, when measured "inside" a system, *KE* is *KE*, and it has no mass associated with it. The proposed definition tells us that the mass of any object is its mass at rest. All else constant, the mass of a cannonball is the same before as after it is fired; mass is invariant. Accordingly, when measured, a photon is massless, even though its *KE* can contribute to the net mass of some encompassing system. This subtlety has led to a fair amount of confusion. As it must, the suggested definition embraces those systems containing massless entities, each of whose *KE* nonetheless can contribute to the overall mass of the system. This is an entirely relativistic phenomenon that has no classical counterpart.

To see how all of this comes together, consider an isolated neutral pion of mass (m_π). The frame in which the system, the pion, does not move is the frame of the pion itself. A pi-zero can decay into two oppositely directed photons each of frequency ν. Viewed from "outside," the net rest energy of the initial state of the system, the pion, is $m_\pi c^2$. The rest energy of the post-decay two-photon state is $2h\nu$ and the system's mass equals $2h\nu/c^2$. Moreover $2h\nu/c^2 = m_\pi$ since ***the total energy of an isolated system is conserved***. Because energy is conserved, when a zero-momentum frame exists, $E = E_0 = mc^2$ and mass is necessarily conserved. As Einstein pointed out, those two conservation laws are no longer independent. Viewed from outside, the pion→photon system conserves mass, and conforms to the above definition; the amount of energy needed to create the system at rest is unaltered by the transformation, hence its mass is unaltered. Viewed from inside, the pion vanishes along with m_π, two massless photons appear, and mass is converted into *KE*. That's also in accord with the above definition since the pion's mass was m_π, and the photons are individually massless.

Because internal *KE* can contribute to the net mass of a composite system, it is possible, even as the mass of the whole remains constant, for the sum of the individual masses of its constituents to decrease (or increase) while their net *KE* increases (or decreases) proportionately. That's what happens when any kind of bomb detonates. *Indeed, in all processes involving an entity possessing mass (be it a stick of dynamite, a candle, or a pion), some portion of the mass of the constituents of the system is converted into KE and/or vice versa.*

When dynamite explodes, internal chemical potential energy arising from the electromagnetic interaction, which manifests itself as mass, is converted into *KE*. If we could collect, *at rest*, all the fragments (i.e., pieces having mass) after the explosion, they would have less net mass than there was to start with. And that would be by an amount equivalent to the liberated *KE* (some of which was associated with photons); that's what powers the explosion. Mass is converted into *KE*. Still, when the system, post explosion, encompasses all the moving fragmentary matter, the associated internal *KE* (including that of photons) contributes to the overall rest energy and leaves the mass of the system—the bomb—unchanged. In other words, viewed from outside, the mass of the system is conserved. And that conforms to the above definition because the explosion does not alter the amount of energy required to create the system, it simply changes the form in which that energy is manifested. Conversely, when two billiard balls slam into each other and come to rest, treating the balls individually we can certainly say that *KE* is converted into mass. That's true even though the mass of the isolated two-ball system, by definition, doesn't change.

Cosmologists tell us that most of the energy of the early universe (the ultimate isolated system) was at one time associated with a blazing maelstrom of massless photons. Today, nearly 13.7 billion years later, the masses of stars and cars and cannonballs are the historical record of eons of transformation and change, and that too accords with our definition.

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7. S. Dürr et al., "Ab initio determination of light hadron masses," *Sci.* **322**, 1224–1227 (Nov. 2008).
8. It is now common practice in quantum mechanics (e.g., when dealing with a Hamiltonian) and in relativity (see, e.g., Ref. 3, p. 631, Eq. 5.1, or Ref. 9, p. 44, Eq. 4) to write momentum squared as $\mathbf{p}^2$. This is done here to remind the reader that we are working with a vector quantity, and that must be considered during the summation of individual momenta. Thus, for a system of particles we will be summing $\mathbf{p}_i$ not p_i.
9. H. Kolbenstvedt, "The mass of a gas of massless photons," *Am. J. Phys*. **63**, 44–46 (Jan. 1995).
10. As Maxwell, Fermi, Einstein, and others have maintained, work is the transfer of energy from one system to another through the action of force. It is therefore a tautology to define energy as the ability to do work. Alternatively, there is matter; it interacts and that interaction results in observable change. Energy, which is associated with time, is the scalar measure of change. Momentum, which is associated with space, is the vector measure of change. In relativity the two are related via the energy-momentum four-vector whose magnitude is the invariant mass. Of course, if energy is the ability to do work, and mass is the resistance to a change in motion, $E_0 = mc^2$ make no sense at all.
11. Eugene Hecht, "Energy and change," *Phys. Teach.* **45**, 88–92 (Feb. 2007).
12. Ibid., Ref. 11, footnote 8.
13. The Heisenberg uncertainty principle allows a particle to exist, if only for a limited time, even though it does not conform to Eq. (1). Such a particle is said to be *virtual* as opposed to *real.* The exchange particles that are responsible for interactions are called gauge bosons and they're all virtual. Several of these (gravitons, gluons, and photons) are massless. Virtual photons have energies that are not equal to cp and do not conform to Eq. (1), as do all *real* particles. Indeed any real object must exist in accord with Eq. (5).
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